Doctor derives the scientific knowledge of the human soul — *i. e.*, *in universali* — from the eternal reasons, as the efficient causes of things, as he had taught in quæst. 84, art. 3.

I offer you this brief *resumé* of the Thomistic philosophy, in the hope that it may serve you as a guide in the study of Saint Thomas.

ALGORITHMIC DIVISION IN LOGIC.

BY GEORGE BRUCE HALSTED.

From its very start, logic has been suffering from the mistaken idea that it was actually an account of all the fundamental principles of legitimate inference, of all valid use of the reasoning faculty.

From the shackles of this self-imposed, but never fulfilled requirement it has not yet quite freed itself, and the confusing effects are visible alike in Ueberweg and Jevons. But once recognized that logic is not a branch of psychology, is conversant with classes of *things*, and that point is passed where it could be believed that mathematics was only a developed branch of ordinary logic, or supposed that the more powerful mathematics was trying to show that logic was only a branch of algebra.

In actual reasoning, the mind, far from being confined to the scholastic logic, jumps, climbs, and runs along in accordance with all sorts of principles, various, though valid.

These results, however, may be stated in terms of ordinary logic — that is, in terms of genus and species — of the relations of classes; and from the generality, simplicity, and certainty of this formal logic, it is, even from the new point of view, as worthy as it was ever thought to be of all study; more especially since those who, recognizing the fundamental character of other relations beside that of the simple copula, have worked on the "Logic of Relatives," have not been able as yet, in spite of the fine contributions made by De Morgan, to bring any cosmos out of that chaos.

But the latter's two statements, "first, logic is the only science which has made no progress since the revival of letters; secondly, logic is the only science which has produced no growth of symbols," were neither true after Boole had put to the science his master hand.

A notation analogous to that used in the coördinate, but more highly developed, science of quantity was found to give to the old and new ideas astonishing vigor. Boole summarizes his result by saying: "Let us conceive, then, of an algebra in which the symbols x, y, z, etc., admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes of such an algebra will be identical in their whole extent with the laws, the axioms, and the processes of an algebra of logic." But this statement must be interpreted very narrowly to be at all exact.

That the slightest extension of the analogy to cause or reason must lead us all wrong is evident from the fact that this algebra admits of only two phases, 0 and 1, while logic admits of three phases, namely, not only *none* and *all*, corresponding to 0 and 1, but also *some*, "which, though it may include in its meaning *all*, does not include *none*" (Boole, p. 124), and hence has no analogue in such an algebra. Again, this algebra may, perhaps, be called unduly arithmetical.

From the idea of the convertibility and transitiveness of the relation expressed by the ordinary copula, or from the equal balance of subject and predicate throughout the formal logic of absolute terms, one would look for an exact correspondence of theorems, subject and predicate being transposed.

Now, of the Boolian product we know, besides the peculiar law $xx=x^2=x$, that also xy is either identical with, or less than, either of the factors. This we may write $xy = or < x$, and $xy = or < y$; and if $z = or < x$ and $z = or < y$, then $z = or < xy$.

From the principle of correspondence there would thus be another function, F (xy), such that $x = or < F\ (xy)$, and $y = or < F\ (xy)$, and if $x = or < z$, and $y = or < z$, then $F\ (xy) = or < z$.

This function is logical addition, which we may distinguish

from Boole's by a subscript comma (+,). It must be by a slip that Prof. Jevons, in the preface to the second edition of his Principles of Science, calls it Boole's.

He says (p. xvii) of Leibnitz: "He first gives as an axiom the now well known law of Boole, as follows:

"'Axioma I. Si idem secum ipso sumatur, nihil constituitur novum, seu A+, A=A.'" Now, no one knows better than Prof. Jevons that the way in which Boole entirely avoids this sort of addition, with its accompanying "Law of Unity," is one of the marked peculiarities of his system.

However much this kind of addition seems called for by logical simplicity, by the principle of correspondence, by the balance of multiplication and addition, yet, besides not agreeing with Boole's arithmetical analogy, it has the grave defect of not being an invertible operation.

Says Boole, page 33: "But the very idea of an operation effecting some positive change seems to suggest to us the idea of an opposite or negative operation, having the effect of undoing what the former one has done. Thus, we cannot conceive it possible to collect parts into a whole, and not conceive it also possible to separate a part from a whole." It is very true that in treating certain subjects — as, for example mathematics — great advantage arises from the fact that you are able to use invertible addition and multiplication, your subtraction and division being determinative.

But in this case, though if b +, x = a, then x = a — b, yet is x not completely determinate. It may vary from *a* to *a* with *b* taken away. The noting of this peculiar fact led Prof. Jevons, in 1864, in his "Pure Logic," to say, page 80: "But addition and subtraction do not exist, and do not give true results, in a system of pure logic, free from the condition of number. For instance, take the logical proposition A +, B+, C = A +, D +, E meaning *what is either A or B or C is either A or D or E, and vice versâ.* In these circumstances, the action of subtraction does not apply. It is not necessarily true that, if from same (equal) things we take same (equal) things the remainders are same (equal). It is not allowable for us to subtract the same thing (A) from both sides of the above

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proposition, and thence infer B +, C = D +, E. This is not true if, for instance, each of B and C is the same as E, and D is the same as A, which has been taken away."

This last sentence is very true, but it does not prove his statement, much less does it warrant his saying, as he does, on the next page, "The axioms of *addition* and subtraction," etc., for you may always logically *add* as many terms as you choose to both sides with perfect safety. He has also failed to notice that by parity of reasoning he must sweep away logical division, which corresponds to Abstraction, but which he calls "Separation," devoting to it chapter V. For, denoting logical division by (;), if bx = a, then x = a ; b. But it will be observed that x is not fully determined by this condition. It will vary from a to $a + \bar{b}$, and will be uninterpretable if a is not wholly contained under b. This only shows that logical multiplication is not invertible; and though Boole was able to make addition invertible and arithmetical by convening that the sign + should only appear between terms mutually exclusive, yet even he failed in regard to logical division, and bolstering himself by what I have shown in a previous paper to be an erroneous analogy, he left his system straddling the fence, having one of the fundamental operations (+) invertible and the other (×) not. He says, page 36: "Hence it cannot be inferred from the equation zx = zy that the equation x = y is also true. In other words, the axiom of the algebraists that both sides of an equation may be divided by the same quantity has no formal equivalent here." In the article on "Boole's Logical Method," I showed how this follows necessarily from the peculiar sliding sort of multiplication found in logic, where if one factor is wholly or in part identical with another, we have an analogy to the fact that superimposing mathematical planes does not increase the thickness, or the one may slide wholly or partly into the other and leave no trace.

I there gave an example, using purposely terms whereof one "rational" is part of the meaning of another "man."

Let us now add the consideration of an example where this is not the case.

Suppose x, y, and z to be none of them included in each other, and that $zx = zy$, which interpret, stratified rocks = rocks deposited from water.

We cannot divide out the common term leaving stratified things = things deposited from water, because the proposition, in the positive information which it gives about zx and zy, conveys nothing about the relation of $x\bar{z}$ to $y\bar{z}$.

If we could only legitimately conclude $\bar{z}x = \bar{z}y$, then we might safely divide and say $x = y$.

An eminent author wrote me as his opinion that the proposition gave no information "about $\bar{x}z$ or $\bar{y}z$ (unstratified rocks, or rocks not deposited from water)." This was probably only a momentary slip, but it leads me to call attention to the fact that the proposition does tell us $\bar{x}z = \bar{y}z$, *i. e.*, unstratified rocks = rocks not deposited from water; but this is of no help to us in rendering division possible.

We certainly can not in any off-hand way, or without the introduction of absurd terms similar to the imaginary in common algebra, make our logical multiplication throughout simply invertible.

But if we could exchange +, and $\times$ for two invertible processes, and thus avoid the incongruity of Boole's system, would we not, after all, still be sacrificing logical simplicity in the real analysis and analogies of the subject to desired ease of a working calculus?

Inverse operations are defined from the direct. A logical quotient, then, is the solution of the equation $xb = a$ . . . (1) in respect to x. This we have already denoted by $x = a ; b$ . . . (2), and noted that the solution is indefinite.

But it is very remarkable that in this expression, independently of the value of x, the classes *a* and *b* cannot be taken arbitrarily; for the equation $bx = a$ involves an independent relation between the classes *a* and *b*, namely, $a\bar{b} = o$ . . . (3) which we may obtain by eliminating x, without regard to its value. We see from this that division in logic is by no means an unrestrictedly practicable operation, and to fully replace (1), we must have not only (2), but also (3).

This equation (3) is the necessary *condition* assumed before

we can talk of the logical quotient of *a* by *b*. a ; b has no sense unless this requirement is fulfilled. Whenever we speak of a quotient we assume this.

Now, for the value of $x = a ; b$, we have

$$a ; b = (a +, \bar{b})\ (v +, b), \text{ or}$$
$$= a +, v\bar{b}, \text{ or}$$
$$= ab +, v\,\overline{ab}.$$

Where v is an arbitrary, an indefinite class. $o ; o = v$.

By the use of this v the above equations for a ; b contain all the particular solutions which arise when the real value of x is more definitely fixed or known.

Two cases are especially worthy of notice: the widest where $v = 1$ (the universe), and the narrowest where $v = o$.

In the latter case the quotient is seen to be coincident with the dividend, *a*. In the former, the maximum case, $a : b = a + \bar{b}$. If here we take $a = o$, we have $o : b = \bar{b} = 1 - b$. Here the condition $a\bar{b} = o$ becomes a mere identity, and may be neglected, showing that this operation may always be performed. In general, for any product xy, it is immediately allowable if $x +, y = 1$. So if $a = b \therefore a : b = 1$.

To continue on deriving division formulæ in this remarkable algebra is an exercise highly suggestive and interesting, but in reality in the above special case, $o : b = 1 - b$, we have all that is necessary for a solution of the logical problem.

It amounts simply to the old familiar operation of forming the negative of a term, and together with +, and $\dot{\times}$ gives in the simplest possible way all the deductive powers attained by Boole's complicated and ill-balanced, yet wonderful, calculus.

Moreover, in reference to these operations, the existence of a perfect *duality* enables the whole matter, like modern geometry, to be exhibited in pairs of corresponding theorems:

e. g., I. $aa = a$. I′. $a +, a = a$.

II. $a\,(b +, c) = ab +, ac$. II′. $a +, bc = (a +, b)\ (a +, c)$.

As a final recommendation, uninterpretable steps are thus entirely obviated, each step being susceptible of simple statement in the ordinary language of logic.

This rounded system, expanded so as to be easily understood by beginners, will be called Dual Logic.